AMERICAN UNIVERSITY OF BEIRUT
MATHEMATICS 218
FALL SEMESTER 2009-2010
FINAL EXAMINATION

Time: 120 minutes Date: January 27, 2010

Name:..

Solution

ID:..

Circle your section number in the table below:

Instructors	Azar	Egeileh	El Khoury	Fuleihan	Karam	Nassif
Section	8	5	1	6	4	2
Section			3		9	7

QUESTION	GRADE
1	/25
2	/25
3	/25
4	/26
5	/30
6	/34
7	/35
TOTAL GRADE	/200

Answer the following seven sets of questions; the back of pages may be used as scratch.

1. Let $T : R^3 \to R^4$ be defined by:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y+3z \\ x+2y \\ y+z \\ z \end{bmatrix}$$

(a) Show that T is a linear transformation. (8 points)
(b) Show that T is one-to-one. (6 points)
(c) What is the rank of the matrix representing T. (4 points)
(d) Find a basis for the range of T. (7 points)

(a) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} 2x+y+3z \\ x+2y \\ y+z \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

T is a matrix multiplication by the 4×3 matrix A.
T is therefore a linear transformation.

Or

For all $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ in $\mathbb{R}^3$, and for all $k \in \mathbb{R}$,

$u + kv = \begin{pmatrix} x+kx' \\ y+ky' \\ z+kz' \end{pmatrix}$ and $T(u+kv) = \begin{pmatrix} 2(x+kx') + (y+ky') + 3(z+kz') \\ (x+kx') + 2(y+ky') \\ (y+ky') + (z+kz') \\ (z+kz') \end{pmatrix}$

yielding $T(u+kv) = \begin{pmatrix} 2x+y+3z \\ x+2y \\ y+z \\ z \end{pmatrix} + k \begin{pmatrix} 2x'+y'+3z' \\ x'+2y' \\ y'+z' \\ z' \end{pmatrix} = T(u) + kT(v)$.

Hence the linearity of T.

(b) Find the nullspace of T:

$\forall v \in N(T)$, $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and is such that $T(v) = 0 \iff \begin{pmatrix} 2x+y+3z \\ x+2y \\ y+z \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

It follows: $\begin{cases} 2x+y+3z = 0 \\ x+2y = 0 \\ y+z = 0 \\ z = 0 \end{cases}$ which is a homogeneous linear system having only the trivial solution $x=y=z=0$

Therefore $N(T) = \{0\} \subset \mathbb{R}^3 \iff T$ is one-to-one.

(c) Find a row-echelon form of A:

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 0 & 3/2 & -3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of the matrix A, representing the linear transformation T, is equal to the common dimension of $col(A)$ and $row(A)$. Since the obtained row-echelon form has 3 leading entries, one concludes that $\text{rank}(A) = 3$.

(d) Range of T: $R(T) = col(A)$ (T being a matrix multiplication by A)

Basis for $R(T)$: $B = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ (columns of A corresponding to the leading entries of its Row-echelon form)

2. Let $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 3 \\ 3 & 5 & -6 & -5 \end{pmatrix}$

(a) Find a basis for the $N(A)$. (10 points)

(b) Determine a basis for $N(A)^{\perp}$ with respect to the Euclidean inner product. (7 points)

(c) What is the dimension of $N(A^T)$? (8 points)

(a) $N(A) = \{x \in \mathbb{R}^4 ,\ Ax = 0\}$

$Ax = 0 \iff \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 3 \\ 3 & 5 & -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Row-echelon form of A:

$A \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & -3 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & -1 & 1 \end{pmatrix} = R$

$\therefore\ Ax = 0 \iff \begin{cases} x_1 + 2x_2 - x_3 + x_4 = 0 \\ -x_2 - 3x_3 - 8x_4 = 0 \\ -x_3 + x_4 = 0 \end{cases}$

let $x_4 = t$
then $x_3 = t$
$x_2 = -11t$
$x_1 = 22t$

$x \in N(A) \iff x = t\begin{pmatrix} 22 \\ -11 \\ 1 \\ 1 \end{pmatrix}_{t \in \mathbb{R}}$ yielding $\boxed{N(A) = \text{Span}\left\{\begin{pmatrix} 22 \\ -11 \\ 1 \\ 1 \end{pmatrix}\right\}}$ $\longrightarrow v_1 \in \mathbb{R}^4$

$B = \{v_1\}$ is a basis for $N(A)$.

(b) The orthogonal complement of $N(A)$ is the row space of A:

$N(A)^{\perp} = \text{row}(A)$.

Therefore, a basis for $N(A)^{\perp}$ is a basis for row(A). Such basis could be made of the row of the row-echelon form of A containing the leading entries:

$B' = \{v_2, v_3, v_4\}$ where $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ -1 \\ -3 \\ -8 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

(c) size $(A) = 3 \times 4$
size $(A^t) = 4 \times 3$
Therefore: $\dim(N(A^t)) + \text{rank}(A^t) = 3$

But $\text{rank}(A^t) = \text{rank}(A) = 3$ (common dimension of col (A) and row (A))

It follows: $\dim N(A^t) = 0$

or equivalently $\text{nullity}(A^t) = 0$

3. Let $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

(*a*) Find the eigenvalues of A. (10 points)
(*b*) Find P that diagonalizes A. (10 points)
(*c*) Deduce $P^{-1}AP$. (5 points)

(a) The eigenvalues of A are the solutions of the characteristic equation: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$$ — Expanding along the first row, it yields:

$$(1-\lambda)[(1-\lambda)^2 - 1] + [\lambda - 1 - 1] - [1 + 1 - \lambda] = 0$$

$$-\lambda(1-\lambda)(2-\lambda) + 2(\lambda - 2) = 0 \iff (\lambda-2)[+\lambda(1-\lambda)+2] = 0$$

$$\iff (\lambda-2)(-\lambda^2+\lambda+2) = 0 \iff \lambda - 2 = 0 \text{ or } -\lambda^2+\lambda+2 = 0$$

$$\underline{\lambda = 2} \qquad \underline{\lambda = -1} \text{ or } \underline{\lambda = 2}$$

(b) Find bases for the eigenspaces of A: (then deduce P)

* Eigenspace corresponding to $\lambda_1 = 2$ (of algebraic multiplicity equal to 2)

This is the solution set of $(A - \lambda_1 I)x = 0$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

makes the linear system equivalent to the unique linear equation: $x_1 + x_2 + x_3 = 0$

Let $x_3 = t$ and $x_2 = s$, then $x_1 = -s - t$

implying $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} = s\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{v_1} + t\underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{v_2}$

$\{v_1, v_2\}$ spans the eigenspace corresponding to $\lambda_1 = 2$ and is linearly independent (making the geometric multiplicity of λ_1 equal to its algebraic one)

$\therefore \{v_1, v_2\}$ is a basis for this eigenspace.

* Eigenspace corresponding to $\lambda_2 = -1$

$(A - \lambda_2 I)x = 0$ $\quad \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3/2 & -3/2 & 0 \\ 0 & -3/2 & 3/2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Equivalent triangular system: $\begin{cases} 2x_1 - x_2 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$

Let $x_3 = t$, then $x_2 = t$ and $x_1 = t$ implying $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{v_3}$

$\{v_3\}$ is a basis for the eigenspace corresponding to $\lambda_2 = -1$

* $P = (v_1 | v_2 | v_3) = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ diagonalizes A,

(c) and $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

4. (*a*) Without solving, find condition(s) on α and β for the following system to have exactly one solution: (10 points)

$$\begin{cases} x+y+\alpha z = 1 \\ x+y+\beta z = 2 \\ \alpha x+\beta y+z = -1 \end{cases}$$

(*b*) For $\alpha = \beta = 0$ find the least squares solution(s) of the linear system. (16 points)

(a) The given linear 3x3 system has a unique solution iif the coefficient matrix is invertible, that is iif its determinant is nonzero:

$$\begin{vmatrix} 1 & 1 & \alpha \\ 1 & 1 & \beta \\ \alpha & \beta & 1 \end{vmatrix} \neq 0 \iff [1-\beta^2] - [1-\alpha\beta] + \alpha[\beta-\alpha] \neq 0$$

$$\iff -\alpha^2 - \beta^2 + 2\alpha\beta \neq 0$$

$$\iff -(\alpha-\beta)^2 \neq 0 \iff \boxed{\alpha \neq \beta}$$

$$\iff \alpha - \beta \neq 0$$

(b) For $\alpha = \beta = 0$, the linear system becomes: $\begin{cases} x+y = 1 \\ x+y = 2 \\ z = -1 \end{cases}$

and is obviously inconsistent.

Its matrix form is: $\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}_{b}$

Any least squares solution to this linear system is the exact solution to the corresponding normal system: $\boxed{A^tAX = A^tb}$ (Here $A^t = A$)

Find A^tA:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A^tA}$$

and A^tb:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}}_{A^tb}$$

Row-reducing the augmented matrix of $A^tAX = A^tb$:

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & 3 \\ 2 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

yields the equivalent triangular linear system:

$$\begin{cases} 2x_1 + 2x_2 = 3 \\ x_3 = -1 \end{cases}$$

let $x_2 = t$

then $x_1 = \frac{1}{2}(3-2t) = \frac{3}{2} - t$

Hence, the given system has infinitely many least squares solutions of the form:

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} - t \\ t \\ -1 \end{pmatrix}} = \begin{pmatrix} 3/2 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{where } t \in \mathbb{R}$$

5. Let $W = \{p(x) \in P_3 /\ xp(2) - p(1) = 0\}$

(a) Prove that W is a subspace of P_3. (8 points)
(b) Find a basis for W. (12 points)
(c) Let $T : P_3 \to P_1$ be the linear transformation defined by:

$$T(p(x)) = xp(2) - p(1)$$

Show that T is onto. (10 points)

(a) W subspace of P_3

$\forall p, q \in W$, $\forall k \in \mathbb{R}$,

$$x[(p+kq)(2)] - (p+kq)(1) = x[p(2) + kq(2)] - p(1) - kq(1)$$
$$= [xp(2) - p(1)] + k[xq(2) - q(1)]$$
$$= 0 \quad \text{since } p, q \in W$$

Hence, $(p+kq) \in W$ meaning that W is closed under addition and scalar multiplication and therefore a subspace of P_3.

(b) Basis for W; Find first vectors (polynomials) spanning W.

$\forall p \in W$, $p(x)$ is of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

with $p(2) = a_0 + 2a_1 + 4a_2 + 8a_3$
$p(1) = a_0 + a_1 + a_2 + a_3$
and $xp(2) - p(1) = 0$ (zero-polynomial) $(\forall x \in \mathbb{R})$
i.e. $(a_0 + 2a_1 + 4a_2 + 8a_3)x - (a_0 + a_1 + a_2 + a_3) = 0$
using the identity property: $\begin{cases} a_0 + a_1 + a_2 + a_3 = 0 \\ a_0 + 2a_1 + 4a_2 + 8a_3 = 0 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 8 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 7 & 0 \end{array}\right)$$

$\Longleftrightarrow \begin{cases} a_0 + a_1 + a_2 + a_3 = 0 \\ a_1 + 3a_2 + 7a_3 = 0 \end{cases}$ yields $a_1 = -3s - 7t$ and $a_0 = 2s + 7t$

Setting $a_3 = t$ and $a_2 = s$ yields $p(x) = (2s+7t) + (-3s-7t)x + sx^2 + tx^3$.

Therefore, $p(x) = (2s + 7t) + (-3s-7t)x + sx^2 + tx^3$
$= s(2 - 3x + x^2) + t(7 - 7x + x^3)$

Let $b_1(x) = 2 - 3x + x^2$ and $b_2(x) = 7 - 7x + x^3$.

Then, $\forall p \in W$, $p = sb_1 + tb_2$ where $s, t \in \mathbb{R}$

i.e. $\{b_1, b_2\}$ span W.

But b_1 and b_2 are linearly independent (distinct degrees)

Therefore $\{b_1, b_2\}$ is a basis for W.

(c) $T(p(x)) = xp(2) - p(1)$ $\forall p \in P_3$

T is onto

Note first that $W = N(T)$ (nullspace of T),
since $W = \{p(x) \in P_3 / T(p(x)) = 0\}$

And $\dim(W) = 2$ yielding $\dim(N(T)) = 2$

Knowing that $\dim(N(T)) + \dim(R(T)) = \dim(P_3)$
it follows: $2 + \dim(R(T)) = 4 \Longleftrightarrow \dim(R(T)) = 2$

But $R(T)$ is a subspace of P_1 and $\dim(P_1) = 2 = \dim(R(T))$

Therefore, $R(T) = P_1 \Longleftrightarrow T$ is onto.

6. Let P_2 be the set of polynomials of degree less or equal than 2. Define the inner product on P_2 by:

$$\boxed{<p,q>= a_0b_0 + a_1b_1 + a_2b_2}$$

where $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$.

Let $p(x) = 1 + x$, $q(x) = 1 + 2x^2$ and $W = \text{span}\,\{p, q\}$

(a) Find λ such that $(q - \lambda p)$ and p are orthogonal. (6 points)
(b) Show that $\{p, q\}$ is a basis for W. (4 points)
(c) Use the Gram-Schmidt process to find an orthogonal basis for W. (10 points)
(d) Find a basis for $W^\perp$. (10 points)
(e) Deduce an orthogonal basis for P_2. (4 points)

(a) $\underline{(q-\lambda p) \text{ and } p \text{ are orthogonal}} \iff \langle q-\lambda p, p\rangle = 0$

$\iff \langle q,p\rangle - \lambda\langle p,p\rangle = 0$

with $\left.\begin{array}{l} p(x) = 1+x \\ q(x) = 1+2x^2 \end{array}\right\}$ yielding: $\begin{array}{l} \langle p,q\rangle = 1 \\ \langle p,p\rangle = 2 = \|p\|^2 \end{array}$

λ should then satisfy: $1 - 2\lambda = 0 \iff \underline{\lambda = 1/2}$

(b) $\underline{\{p,q\} \text{ basis for } W = \text{span}\{p,q\}}$

Consider the vector equation: $k_1p + k_2q = 0$

i.e. $\forall x \in \mathbb{R}$, $k_1(1+x) + k_2(1+2x^2) = 0$

$(k_1+k_2) + k_1x + 2k_2x^2 = 0$

It follows: $\left\{\begin{array}{l} k_1 + k_2 = 0 \\ k_1 = 0 \\ 2k_2 = 0 \end{array}\right.$ and therefore $k_1 = k_2 = 0$,

making $\{p, q\}$ linearly independent and therefore a basis for W.

(c) $\underline{\text{Orthogonal basis for } W}$: let $\{b_1, b_2\}$ be such basis.
Using the Gram-schmidt process:

- $b_1 = p = 1+x$
- $b_2 = q - \text{proj}_{b_1} q = q - \dfrac{\langle q,b_1\rangle}{\|b_1\|^2} b_1 = 1+2x^2 - \dfrac{1}{2}(1+x) = \dfrac{1}{2} - \dfrac{1}{2}x + 2x^2$

$\therefore$ $\{b_1, b_2\} = \{1+x,\ \frac{1}{2} - \frac{1}{2}x + 2x^2\}$ is an orthogonal basis for W.

(d) $\underline{\text{Basis for } W^\perp = (\{p,q\})^\perp}$

Since $\{p,q\}$ is a basis for W, then:

$v \in W^\perp \iff [\langle v,p\rangle = 0 \ \underline{\text{and}}\ \langle v,q\rangle = 0]$

Let $v = c_0 + c_1x + c_2x^2$, yielding $\left\{\begin{array}{l} \langle v,p\rangle = c_0 + c_1 \\ \langle v,q\rangle = c_0 + 2c_2 \end{array}\right.$

So $v \in W^\perp \iff \left\{\begin{array}{l} c_0 + c_1 = 0 \\ c_0 + 2c_2 = 0 \end{array}\right.$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right)$ yielding the equivalent system $\left\{\begin{array}{l} c_0 + c_1 = 0 \\ -c_1 + 2c_2 = 0 \end{array}\right.$

Let $c_2 = t$, then $c_1 = 2t$ and $c_0 = -2t$.

So that: $v \in W^\perp \iff v = -2t + 2tx + tx^2 = t\underbrace{(-2+2x+x^2)}_{b_3(x)}$

$\underline{\{b_3\} = \{-2+2x+x^2\}}$ is a basis for $W^\perp$

(e) $b_3 \in W^\perp$ and $\{b_1, b_2\}$ is an orthogonal basis for W with $\langle b_1, b_2\rangle = 0$
then $\langle b_3, b_1\rangle = 0$ and $\langle b_3, b_2\rangle = 0$,
It follows that $\underline{\{b_1, b_2, b_3\} = \{1+x,\ \frac{1}{2} - \frac{1}{2}x + 2x^2,\ -2+2x+x^2\}}$ is an orthogonal basis for $P_2 = W \oplus W^\perp$.

7. Prove the following statements:

(a) $T : V \to V$ such that T is one-to-one, prove that if $\{v_1, \ldots, v_n\}$ are linearly independent in V, then $\{T(v_1), \ldots T(v_n)\}$ are linearly independent.

(10 points)

Consider the vector equation:

$$c_1 T(v_1) + \cdots + c_n T(v_n) = 0 \quad \text{) T linear}$$

$$\Rightarrow \quad T(c_1 v_1 + \cdots + c_n v_n) = 0 \quad \text{) T one-to-one}$$

$$\Rightarrow \quad c_1 v_1 + \cdots + c_n v_n = 0 \quad \text{) } \{v_1, \ldots, v_n\} \text{ linearly independent}$$

$$\Rightarrow \quad c_1 = \cdots = c_n = 0$$

Therefore $c_1 T(v_1) + \cdots + c_n T(v_n) = 0$ has only the trivial solution, making $\{T(v_1), \ldots, T(v_n)\}$ linearly independent.

(b) Let V be an inner product space. Show that:

$||u|| = ||v||$ <=> $u + v$ and $u - v$ are orthogonal

(10 points)

<u>$u+v$ and $u-v$ are orthogonal</u> $\Leftrightarrow \langle u+v, u-v \rangle = 0$

$$\Leftrightarrow \langle u, u \rangle + \langle v, u \rangle - \langle u, v \rangle - \langle v, v \rangle = 0$$

$$\Leftrightarrow \|u\|^2 + 0 - 0 - \|v\|^2 = 0$$

$$\Leftrightarrow \|u\|^2 = \|v\|^2$$

$$\Leftrightarrow \boxed{\|u\| = \|v\|}$$ since norms of vectors are positive numbers.

(c) Let B be an $n \times n$ invertible matrix and $T : M_{n\times n} \to M_{n\times n}$ be defined by $T(A) = AB$. Show that T is an isomorphism. (15 points)

* T is a linear transformation:

$\forall A_1, A_2 \in M_{n\times n}$, $\forall k \in \mathbb{R}$,

$$T(A_1 + kA_2) = (A_1 + kA_2)B = A_1B + kA_2B = T(A_1) + kT(A_2)$$

Hence the linearity of T

* T is one-to-one: Let N(T) be the nullspace of T.

$\forall A \in N(T)$, $T(A) = 0 \iff AB = 0$

but B is invertible, so $ABB^{-1} = 0B^{-1}$

$\boxed{A = 0}$ (zero-matrix)

$\therefore N(T) = \{0\}$ making T one-to-one.

* T is onto:

$\forall M \in M_{n\times n}$, if there exists a matrix A such that $T(A) = M$, then $\left.\begin{array}{l} AB = M \\ B \text{ invertible} \end{array}\right\} \Rightarrow A = MB^{-1}$

$\therefore$ Every matrix M has a pre image in $M_{n\times n}$ which is $A = MB^{-1}$

* Therefore, T is an isomorphism.